

# Uncertainty functions of the open quantum harmonic oscillator in the Lindblad theory

A. Isar  $\dagger$ <sup>(a)</sup> and W. Scheid  $\ddagger$

$\dagger$  *Department of Theoretical Physics, Institute of Atomic Physics,  
Bucharest-Magurele, Romania*

$\ddagger$  *Institut für Theoretische Physik der Justus-Liebig-Universität,  
Giessen, Germany*

## Abstract

In the framework of the Lindblad theory for open quantum systems, we derive closed analytical expressions of the Heisenberg and Schrödinger generalized uncertainty functions for a particle moving in a harmonic oscillator potential. The particle is initially in an arbitrary correlated coherent state, and interacts with an environment at finite temperature. We describe how the quantum and thermal fluctuations contribute to the uncertainties in the canonical variables of the system and analyze the relative importance of these fluctuations in the evolution of the system. We show that upon contact with the bath the system evolves from a quantum-dominated to a thermal-dominated state in a time that is of the same order as the decoherence time calculated in other models in the context of transitions from quantum to classical physics.

PACS numbers: 03.65.Ta, 05.30.-d, 05.40.-a

(a) e-mail address: isar@theory.nipne.ro

## 1 Introduction

The present work is directed towards the study of the Heisenberg and Schrödinger generalized uncertainty functions and the role of quantum and thermal fluctuations during the evolution of a system consisting of a particle moving in a harmonic oscillator potential and interacting with an environment. There is a large amount of papers concerned with the question of modification and generalization of the uncertainty principle, in particular on modified uncertainty relations which incorporate the effect of environmentally induced fluctuations [1, 2, 3]. The uncertainty relations derived in [1] are information-theoretic relations in terms of the Shannon information (Wehrl entropy). In Ref. [2], Anastopoulos and Halliwell derived such relations for the Heisenberg uncertainty function  $U = \sigma_{qq}\sigma_{pp}$  and for the Schrödinger generalized uncertainty function  $\sigma = \sigma_{qq}\sigma_{pp} - \sigma_{pq}^2$ . Here  $\sigma_{qq}$ ,  $\sigma_{pp}$  and  $\sigma_{pq}$  denote the variances and, respectively, the covariance of coordinate and momentum of the open system.

In a series of papers Hu and Zhang computed the time evolution of the Heisenberg uncertainty function  $U$  in the presence of a thermal environment. They noted for the first time the significance of the decoherence time scale for quantum and thermal fluctuations of similar sizes [4, 5]. All these authors used quantum Brownian models consisting of a particle moving in a potential and linearly coupled to a bath of harmonic oscillators in a thermal state. Their results show that in the Fokker-Planck regime, decoherence and thermal fluctuations become important on the same time scale.

In this paper we consider Heisenberg and Schrödinger generalized uncertainty functions for a harmonic oscillator coupled with an environment. Our model is elaborated in the framework of the Lindblad theory for open systems.

It is generally thought that quantum dynamical semigroups are the basic tools to introduce dissipation in quantum mechanics [6, 7, 8]. In Markovian approximation and for weakly damped systems, the most general form of the generators of such semigroups was given by Lindblad [9]. This formalism has been studied extensively for the case of damped harmonic oscillators [8, 10, 11, 12, 13] and applied to various physical phenomena, for instance, to the damping of collective modes in deep inelastic collisions in nuclear physics [14]. A phase space representation for the open quantum systems within the Lindblad theory was given in Refs. [15, 16].

The paper is organized as follows. In Sec. 2 we remind the basic results concerning the evolution of the damped harmonic oscillator in the Lindblad theory for open quantum systems. Then in Sec. 3 we derive closed analytical expressions for the finite temperature Heisenberg and Schrödinger generalized uncertainty functions. We consider the general case of a thermal bath and take the correlated coherent and squeezed states, in particular the coherent states, as initial states. We consider the limiting cases of both zero and high temperatures of the environment and the limit of short and long times. In Sec. 4 we discuss the relative importance of quantum and thermal fluctuations in the evolution of the system towards equilibrium with the aim of clarifying the meaning of quantum, classical and thermal regimes, motivated by the necessity of understanding the process of decoherence via interaction with the environment and the general problem of the transition from quantum to classical behaviour. In Sec. 5 we discuss our results in connection with other work. A summary and concluding remarks are given in Sec. 6.

## 2 Lindblad master equation for damped harmonic oscillator

The simplest dynamics for an open system which describes an irreversible process is a semigroup of transformations introducing a preferred direction in time [6, 7, 9]. In Lindblad's axiomatic formalism of introducing dissipation in quantum mechanics, the irreversible time evolution of the open system is described by the following general quantum Markovian master equation for the density operator  $\rho(t)$  [9]:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar} \sum_j ([V_j \rho(t), V_j^\dagger] + [V_j, \rho(t) V_j^\dagger]). \quad (1)$$

Here  $H$  is the Hamiltonian operator of the system and  $V_j, V_j^\dagger$  are operators on the Hilbert space of the Hamiltonian, which model the environment. The semigroup dynamics of the density operator which must hold for a quantum Markovian process is valid only for the weak-coupling regime. In the case of an exactly solvable model for the damped harmonic oscillator, the two possible operators  $V_1$  and  $V_2$  are taken as linear polynomials in coordinate  $q$  and momentum  $p$  [8, 10, 11] and the harmonic oscillator Hamiltonian  $H$  is chosen of the most general quadratic form

$$H = H_0 + \frac{\mu}{2}(qp + pq), \quad H_0 = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2. \quad (2)$$

With these choices the master equation (1) takes the following form [8, 11]:

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{2\hbar}(\lambda + \mu)[q, \rho p + p \rho] + \frac{i}{2\hbar}(\lambda - \mu)[p, \rho q + q \rho] \\ & - \frac{D_{pp}}{\hbar^2}[q, [q, \rho]] - \frac{D_{qq}}{\hbar^2}[p, [p, \rho]] + \frac{D_{pq}}{\hbar^2}([q, [p, \rho]] + [p, [q, \rho]]). \end{aligned} \quad (3)$$

The quantum diffusion coefficients  $D_{pp}, D_{qq}, D_{pq}$  and the friction constant  $\lambda$  satisfy the following fundamental constraints [8, 11]:  $D_{pp} > 0, D_{qq} > 0$  and

$$D_{pp}D_{qq} - D_{pq}^2 \geq \frac{\lambda^2 \hbar^2}{4}. \quad (4)$$

In the particular case when the asymptotic state is a Gibbs state  $\rho_G(\infty) = e^{-\frac{H}{kT}} / \text{Tr} e^{-\frac{H}{kT}}$ , these coefficients can be written as [8, 11]

$$D_{pp} = \frac{\lambda + \mu}{2} \hbar m \omega \coth \frac{\hbar \omega}{2kT}, \quad D_{qq} = \frac{\lambda - \mu}{2} \frac{\hbar}{m \omega} \coth \frac{\hbar \omega}{2kT}, \quad D_{pq} = 0, \quad (5)$$

where  $T$  is the temperature of the thermal bath. In this case, the fundamental constraints are satisfied only if  $\lambda > \mu$  and

$$(\lambda^2 - \mu^2) \coth^2 \frac{\hbar \omega}{2kT} \geq \lambda^2. \quad (6)$$

We notice from this relation that in the particular case of  $T = 0$  we must consider  $\mu = 0$ .

Lindblad has proven [10] that in the Markovian regime the harmonic oscillator master equation which satisfies the complete positivity condition cannot satisfy simultaneously the translational invariance and the detailed balance (which assures an asymptotic approach to the canonical thermal equilibrium state). The necessary and sufficient condition for translational invariance is  $\lambda = \mu$  [8, 10, 11]. In the following general values  $\lambda \neq \mu$  will be considered. In this way we violate translational invariance, but we keep the canonical equilibrium state.

By using the complete positivity property of the Lindblad model, in Refs. [8, 11] the following inequality was obtained for all values of  $t \geq 0$ :

$$D_{pp}\sigma_{qq}(t) + D_{qq}\sigma_{pp}(t) - 2D_{pq}\sigma_{pq}(t) \geq \frac{\hbar^2\lambda}{2}. \quad (7)$$

The inequality (7) represents a restriction connecting the values of the variances and covariance with the environment coefficients. In the case of a thermal bath, when the environment coefficients have the form given by Eqs. (5), the condition (7) becomes

$$[(\lambda + \mu)m\omega\sigma_{qq}(t) + (\lambda - \mu)\frac{\sigma_{pp}(t)}{m\omega}] \coth \frac{\hbar\omega}{2kT} \geq \hbar\lambda. \quad (8)$$

We have found in Ref. [12] that the inequality (7) is equivalent with the generalized uncertainty relation at any time  $t$

$$\sigma(t) \equiv \sigma_{qq}(t)\sigma_{pp}(t) - \sigma_{pq}^2(t) \geq \frac{\hbar^2}{4}, \quad (9)$$

if the initial values  $\sigma_{qq}(0)$ ,  $\sigma_{pp}(0)$  and  $\sigma_{pq}(0)$  satisfy this inequality for  $t = 0$ . The relation (4) is a necessary condition for the generalized uncertainty inequality (9) to be fulfilled. Schrödinger [17] and Robertson [18] proved for the operators of coordinate  $q$  and momentum  $p$  that the inequality (9) is in general fulfilled. The equality in relation (9) is realized for a special class of pure states, called correlated coherent states [19] or squeezed coherent states.

The terms in Eq. (3) containing  $\lambda$  and  $\mu$  are dissipative terms. They cause a contraction of each volume element in phase space. The diffusive terms containing  $D_{pp}$ ,  $D_{qq}$  and  $D_{pq}$  produce an expansion of the volume elements and they are responsible for noise and also for the destruction of interference (decoherence). Although the diffusion in general increases the uncertainty  $\sigma(t)$  as time goes on, there are competing effects that may reduce it. In particular, wave packet reassembly (the time reverse of

wave packet spreading) and dissipation may cause the uncertainty to decrease, but relation (9) is always obeyed.

From the master equation (3) we can obtain the equations of motion for the variances and covariance of coordinate and momentum [8, 11] which are needed to calculate the uncertainty functions:

$$\frac{d\sigma_{qq}(t)}{dt} = -2(\lambda - \mu)\sigma_{qq}(t) + \frac{2}{m}\sigma_{pq}(t) + 2D_{qq}, \quad (10)$$

$$\frac{d\sigma_{pp}(t)}{dt} = -2(\lambda + \mu)\sigma_{pp}(t) - 2m\omega^2\sigma_{pq}(t) + 2D_{pp}, \quad (11)$$

$$\frac{d\sigma_{pq}(t)}{dt} = -m\omega^2\sigma_{qq}(t) + \frac{1}{m}\sigma_{pp}(t) - 2\lambda\sigma_{pq}(t) + 2D_{pq}. \quad (12)$$

In this paper we consider the underdamped case ( $\omega > \mu$ ). Introducing the notations

$$X(t) = \begin{pmatrix} m\omega\sigma_{qq}(t) \\ \sigma_{pp}(t)/m\omega \\ \sigma_{pq}(t) \end{pmatrix}, \quad D = \begin{pmatrix} 2m\omega D_{qq} \\ 2D_{pp}/m\omega \\ 2D_{pq} \end{pmatrix}, \quad (13)$$

the solutions of these equations of motion can be written in the form [8, 11]

$$X(t) = (Te^{Kt}T)(X(0) - X(\infty)) + X(\infty), \quad (14)$$

where the matrices  $T$  and  $K$  are given by (we introduce the notation  $\Omega^2 = \omega^2 - \mu^2$ )

$$T = \frac{1}{2i\Omega} \begin{pmatrix} \mu + i\Omega & \mu - i\Omega & 2\omega \\ \mu - i\Omega & \mu + i\Omega & 2\omega \\ -\omega & -\omega & -2\mu \end{pmatrix}, \quad K = \begin{pmatrix} -2(\lambda - i\Omega) & 0 & 0 \\ 0 & -2(\lambda + i\Omega) & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \quad (15)$$

and

$$X(\infty) = -(TK^{-1}T)D. \quad (16)$$

Formula (16) gives a simple connection between the asymptotic values ( $t \rightarrow \infty$ ) of  $\sigma_{qq}(t), \sigma_{pp}(t), \sigma_{pq}(t)$  and the diffusion coefficients  $D_{pp}, D_{qq}, D_{pq}$ . These asymptotic values do not depend on the initial values  $\sigma_{qq}(0), \sigma_{pp}(0), \sigma_{pq}(0)$  and in the case of a thermal bath with coefficients (5), they reduce to [8, 11]

$$\sigma_{qq}(\infty) = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2kT}, \quad \sigma_{pp}(\infty) = \frac{\hbar m\omega}{2} \coth \frac{\hbar\omega}{2kT}, \quad \sigma_{pq}(\infty) = 0. \quad (17)$$

### 3 Heisenberg and Schrödinger uncertainty functions for Lindblad model

#### 3.1 Choice of initial wave function

We consider a harmonic oscillator with an initial wave function

$$\Psi(x) = \left(\frac{1}{2\pi\sigma_{qq}(0)}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{4\sigma_{qq}(0)}\left(1 - \frac{2i}{\hbar}\sigma_{pq}(0)\right)(x - \sigma_q(0))^2 + \frac{i}{\hbar}\sigma_p(0)x\right], \quad (18)$$

where  $\sigma_{qq}(0)$  is the initial spread,  $\sigma_{pq}(0)$  the initial covariance, and  $\sigma_q(0)$  and  $\sigma_p(0)$  are the averaged initial position and momentum of the Gaussian wave packet. As will be seen, the parameters  $\sigma_q$  and  $\sigma_p$  do not appear in the uncertainty functions. As initial state we take a correlated coherent state [19] which is represented by the Gaussian wave packet (18) in the coordinate representation with the variances and covariance of coordinate and momentum

$$\sigma_{qq}(0) = \frac{\hbar\delta}{2m\omega}, \quad \sigma_{pp}(0) = \frac{\hbar m\omega}{2\delta(1-r^2)}, \quad \sigma_{pq}(0) = \frac{\hbar r}{2\sqrt{1-r^2}}. \quad (19)$$

Here,  $\delta$  is the squeezing parameter which measures the spread in the initial Gaussian packet and  $r = r(0)$ ,  $|r| < 1$  is the correlation coefficient at time  $t = 0$ . The correlation coefficient is defined as

$$r(t) = \frac{\sigma_{pq}(t)}{\sqrt{\sigma_{qq}(t)\sigma_{pp}(t)}}. \quad (20)$$

The initial values (19) correspond to a so-called minimum uncertainty state, since they fulfil the generalized uncertainty relation with equal sign

$$\sigma(0) \equiv \sigma_{qq}(0)\sigma_{pp}(0) - \sigma_{pq}^2(0) = \frac{\hbar^2}{4}. \quad (21)$$

For  $\delta = 1$  and  $r = 0$  the correlated coherent state becomes a Glauber coherent state. With the initial values (19) and by assuming a thermal bath with temperature  $T$ , the condition (8) for  $t = 0$  takes the form ( $\epsilon \equiv \hbar\omega/2kT$ ):

$$[(\lambda + \mu)\delta + (\lambda - \mu)\frac{1}{\delta(1-r^2)}] \coth \epsilon \geq 2\lambda. \quad (22)$$

One can easily show that

$$[(\lambda + \mu)\delta + (\lambda - \mu)\frac{1}{\delta}] \geq 2\sqrt{\lambda^2 - \mu^2}. \quad (23)$$

It follows that if relation (6) is satisfied, then relation (22) is also satisfied (since  $|r| < 1$ ) and, therefore, for a given temperature  $T$  of the bath and for any parameters  $\delta$  and  $r$  the inequality (6) alone determines the range of values of the parameters  $\lambda$  and  $\mu$ .

### 3.2 Heisenberg uncertainty function at finite temperature

For simplicity we set  $r = 0$  in this Subsection. With the variances given by Eq. (14) we calculate the Heisenberg uncertainty function  $U(t) = \sigma_{qq}(t)\sigma_{pp}(t)$  for finite temperature and obtain:

$$\begin{aligned}
U(t) = & \frac{\hbar^2}{4} \left\{ e^{-4\lambda t} \left[ 1 - \left( \delta + \frac{1}{\delta} \right) \coth \epsilon + \coth^2 \epsilon + \frac{\omega^2}{4\Omega^4} \left[ \omega^2 \left( \delta - \frac{1}{\delta} \right)^2 \sin^2(2\Omega t) \right. \right. \right. \\
& + 2\mu^2 \left[ (\delta - \coth \epsilon)^2 + \left( \frac{1}{\delta} - \coth \epsilon \right)^2 \right] \cos(2\Omega t) (\cos(2\Omega t) - 1) \\
& \left. \left. \left. + 4\mu^2 (\delta - \coth \epsilon) \left( \frac{1}{\delta} - \coth \epsilon \right) (1 - \cos(2\Omega t)) \right. \right. \right. \\
& + 2\mu\Omega \left[ (\delta - \coth \epsilon)^2 - \left( \frac{1}{\delta} - \coth \epsilon \right)^2 \right] \sin(2\Omega t) (1 - \cos(2\Omega t)) \left. \right] \\
& + e^{-2\lambda t} \coth \epsilon \left[ \left( \delta + \frac{1}{\delta} - 2\coth \epsilon \right) \frac{\omega^2 - \mu^2 \cos(2\Omega t)}{\Omega^2} + \left( \delta - \frac{1}{\delta} \right) \frac{\mu \sin(2\Omega t)}{\Omega} \right] + \coth^2 \epsilon \right\}. \quad (24)
\end{aligned}$$

This is the first main result of the present paper.

For an initial coherent state ( $\delta = 1$ ), this expression simplifies:

$$\begin{aligned}
U(t) = & \frac{\hbar^2}{4} \left\{ e^{-4\lambda t} (\coth \epsilon - 1)^2 \left[ 1 + \frac{\omega^2 \mu^2}{\Omega^4} (1 - \cos(2\Omega t))^2 \right] \right. \\
& \left. + 2e^{-2\lambda t} \coth \epsilon (1 - \coth \epsilon) \frac{\omega^2 - \mu^2 \cos(2\Omega t)}{\Omega^2} + \coth^2 \epsilon \right\}. \quad (25)
\end{aligned}$$

For  $\mu = 0$ , which is a more natural choice, since it corresponds to the usual form of the oscillator Hamiltonian (2), we obtain from Eq. (24):

$$\begin{aligned}
U(t) = & \frac{\hbar^2}{4} \left\{ e^{-4\lambda t} \left[ 1 - \left( \delta + \frac{1}{\delta} \right) \coth \epsilon + \coth^2 \epsilon + \frac{1}{4} \left( \delta - \frac{1}{\delta} \right)^2 \sin^2(2\omega t) \right] \right. \\
& \left. + e^{-2\lambda t} \coth \epsilon \left( \delta + \frac{1}{\delta} - 2\coth \epsilon \right) + \coth^2 \epsilon \right\}. \quad (26)
\end{aligned}$$

For  $\delta = 1$  this expression takes the form

$$U(t) = \frac{\hbar^2}{4} \{ e^{-2\lambda t} + \coth \epsilon (1 - e^{-2\lambda t}) \}^2. \quad (27)$$

Here the first term is of quantum nature, whereas the second term is of thermal nature. Their contributions to the uncertainty of the system arise from quantum and thermal fluctuations, respectively. This expression is similar to that obtained in Refs. [4, 5], the difference being that in  $\coth \epsilon$  instead of the natural frequency  $\omega$  stays the renormalized frequency.

In the obtained expressions of the uncertainty, besides the parameters  $\lambda, \mu$  and  $\delta$  there are also two factors, time and temperature, which we will consider for different

regimes. For example, in the case of  $T = 0$  ( $\coth \epsilon = 1$ ) we have to take, according to Eq. (6),  $\mu = 0$  and then we obtain from Eq. (26):

$$U_0(t) = \frac{\hbar^2}{4} \left\{ 1 + e^{-4\lambda t} \left[ 2 - \left( \delta + \frac{1}{\delta} \right) + \frac{1}{4} \left( \delta - \frac{1}{\delta} \right)^2 \sin^2(2\omega t) \right] + e^{-2\lambda t} \left[ \left( \delta + \frac{1}{\delta} - 2 \right) \right] \right\}. \quad (28)$$

We see in the last expression that the leading term is given by  $\hbar^2/4$  (the Heisenberg contribution) followed, for squeezed states  $\delta \neq 1$ , by terms describing both decay and oscillatory behaviour, representing quantum fluctuations alone (since  $T = 0$ ). For  $\delta = 1$  we obtain in the zero-temperature case  $U_0(t) = \hbar^2/4$ .

### 3.3 Schrödinger uncertainty function at finite temperature

With the variances given by Eq. (14) we calculate now the finite temperature generalized uncertainty function  $\sigma(t) = \sigma_{qq}(t)\sigma_{pp}(t) - \sigma_{pq}^2(t)$  and obtain:

$$\begin{aligned} \sigma(t) = & \frac{\hbar^2}{4} \left\{ e^{-4\lambda t} \left[ 1 - \left( \delta + \frac{1}{\delta(1-r^2)} \right) \coth \epsilon + \coth^2 \epsilon \right] \right. \\ & + e^{-2\lambda t} \coth \epsilon \left[ \left( \delta + \frac{1}{\delta(1-r^2)} - 2 \coth \epsilon \right) \frac{\omega^2 - \mu^2 \cos(2\Omega t)}{\Omega^2} \right. \\ & \left. \left. + \left( \delta - \frac{1}{\delta(1-r^2)} \right) \frac{\mu \sin(2\Omega t)}{\Omega} + \frac{2r\mu\omega(1-\cos(2\Omega t))}{\Omega^2\sqrt{1-r^2}} \right] + \coth^2 \epsilon \right\}. \end{aligned} \quad (29)$$

This is the second main result of the present paper.

When the correlation coefficient  $r = 0$ , the correlated coherent initial state becomes a pure squeezed state. In this case the expression of the uncertainty function takes the form

$$\begin{aligned} \sigma(t) = & \frac{\hbar^2}{4} \left\{ e^{-4\lambda t} \left[ 1 - \left( \delta + \frac{1}{\delta} \right) \coth \epsilon + \coth^2 \epsilon \right] \right. \\ & + e^{-2\lambda t} \coth \epsilon \left[ \left( \delta + \frac{1}{\delta} - 2 \coth \epsilon \right) \frac{\omega^2 - \mu^2 \cos(2\Omega t)}{\Omega^2} + \left( \delta - \frac{1}{\delta} \right) \frac{\mu \sin(2\Omega t)}{\Omega} \right] + \coth^2 \epsilon \left. \right\}. \end{aligned} \quad (30)$$

For the usual coherent state ( $\delta = 1$ ) the uncertainty function is

$$\begin{aligned} \sigma(t) = & \frac{\hbar^2}{4} \left\{ e^{-4\lambda t} (\coth \epsilon - 1)^2 \right. \\ & + 2e^{-2\lambda t} \coth \epsilon (1 - \coth \epsilon) \frac{\omega^2 - \mu^2 \cos(2\Omega t)}{\Omega^2} + \coth^2 \epsilon \left. \right\}. \end{aligned} \quad (31)$$

All the expressions above simplify in the particular case when  $\mu = 0$ . Namely, in this case the uncertainty function (29) has the form

$$\begin{aligned} \sigma(t) = & \frac{\hbar^2}{4} \left\{ e^{-4\lambda t} \left[ 1 - \left( \delta + \frac{1}{\delta(1-r^2)} \right) \coth \epsilon + \coth^2 \epsilon \right] \right. \\ & + e^{-2\lambda t} \coth \epsilon \left[ \delta + \frac{1}{\delta(1-r^2)} - 2 \coth \epsilon \right] + \coth^2 \epsilon \left. \right\}. \end{aligned} \quad (32)$$

For an initial squeezed state ( $r = 0$ ) and any squeezing coefficient  $\delta$ , we obtain

$$\begin{aligned}\sigma(t) = \frac{\hbar^2}{4} &\{e^{-4\lambda t}[1 - (\delta + \frac{1}{\delta}) \coth \epsilon + \coth^2 \epsilon] \\ &+ e^{-2\lambda t} \coth \epsilon (\delta + \frac{1}{\delta} - 2 \coth \epsilon) + \coth^2 \epsilon\}.\end{aligned}\quad (33)$$

When the initial state is the usual coherent state ( $\delta = 1$ ), we obtain

$$\sigma(t) = \frac{\hbar^2}{4} \{e^{-2\lambda t} + \coth \epsilon (1 - e^{-2\lambda t})\}^2, \quad (34)$$

which is identical with Eq. (27) for  $U(t)$ .

We consider now the particular case when the temperature of the thermal bath is  $T = 0$ . Then we have to set also  $\mu = 0$  (cf. Eq. (6)) and the uncertainty function  $\sigma(t)$  takes the following form:

$$\sigma_0(t) = \frac{\hbar^2}{4} \{1 + (e^{-4\lambda t} - e^{-2\lambda t})[2 - (\delta + \frac{1}{\delta(1-r^2)})]\}. \quad (35)$$

We see that in the last expression the leading term is given by  $\hbar^2/4$  (the Heisenberg contribution) followed by terms representing quantum fluctuations alone (since  $T = 0$ ). Compared to Eq. (28), where these terms describe both decay and oscillating behaviour, in Eq. (35) the terms representing the quantum fluctuations describe only a decay behaviour. When the initial state is the usual coherent state ( $\delta = 1, r = 0$ ), the uncertainty function takes again the most simple form  $\sigma_0(t) = \hbar^2/4$  for all times.

## 4 Transition from quantum mechanics to classical statistical mechanics

(a)  $t = 0$  : When the initially uncorrelated condition is assumed valid, we have  $\sigma(0) = U(0) = \hbar^2/4$ , according to Eq. (21).

(b)  $t \gg \lambda^{-1}$  (very long times):  $\sigma(t)$  and  $U(t)$  are insensitive to  $\lambda, \mu, \delta$  and  $r$  and approach

$$\sigma^{BE} = U^{BE} = \frac{\hbar^2}{4} \coth^2 \epsilon, \quad (36)$$

which is a Bose-Einstein relation for a system of bosons in equilibrium at temperature  $T$  (quantum statistical mechanics). Again  $T = 0$  is the limit of pure quantum fluctuations,

$$\sigma_0 = U_0 = \frac{\hbar^2}{4}, \quad (37)$$

which is the quantum Heisenberg relation and high  $T$  ( $T \gg \hbar\omega/k$ ) is the limit of pure thermal fluctuations,

$$\sigma^{MB} = U^{MB} = \left(\frac{kT}{\omega}\right)^2, \quad (38)$$

which is a Maxwell-Boltzmann distribution for a system approaching a classical limit (in classical statistical mechanics the equipartition theorem imparts for each degree of freedom an uncertainty of  $kT/2$ ). These are expected results from quantum mechanics and classical statistical mechanics. The formula (36) interpolates between the two results (37) at  $T = 0$  and (38) at  $T \gg \hbar\omega/k$ .

(c)  $r = 0$  : At short times ( $\lambda t \ll 1, \Omega t \ll 1$ ), we obtain from Eqs. (24) and (30):

$$\sigma(t) = U(t) = \frac{\hbar^2}{4} \left\{ 1 + 2[\lambda(\delta + \frac{1}{\delta}) \coth \epsilon + \mu(\delta - \frac{1}{\delta}) \coth \epsilon - 2\lambda]t \right\}. \quad (39)$$

The time when thermal fluctuations overtake quantum fluctuations is

$$t_d = \frac{1}{2[\lambda(\delta + \frac{1}{\delta}) \coth \epsilon + \mu(\delta - \frac{1}{\delta}) \coth \epsilon - 2\lambda]}. \quad (40)$$

According to the theory of Halliwell [1, 2] and Hu [4, 5], we expect this time to be equal to the decoherence time scale, which is not yet calculated for the damped harmonic oscillator in the Lindblad model for open quantum systems.

For  $\mu = 0$  we obtain from Eq. (39):

$$\sigma(t) = U(t) = \frac{\hbar^2}{4} \left\{ 1 + 2\lambda[(\delta + \frac{1}{\delta}) \coth \epsilon - 2]t \right\}. \quad (41)$$

In this case

$$t_d = \frac{1}{2\lambda[(\delta + \frac{1}{\delta}) \coth \epsilon - 2]}. \quad (42)$$

For  $\delta = 1$  the uncertainty function (39) is independent of  $\mu$ .

(i) At temperature  $T = 0$  the uncertainty (39) becomes ( $\mu = 0$ )

$$\sigma_0(t) = U_0(t) = \frac{\hbar^2}{4} \left\{ 1 + 2\lambda(\delta + \frac{1}{\delta} - 2)t \right\} \quad (43)$$

and

$$t_d = \frac{1}{2\lambda(\delta + \frac{1}{\delta} - 2)}. \quad (44)$$

(ii) In the case of high temperatures, introducing the notation

$$\tau \equiv \frac{2kT}{\hbar\omega} \equiv \frac{1}{\epsilon}, \quad (45)$$

we obtain

$$\sigma(t) = U(t) = \frac{\hbar^2}{4} \left\{ 1 + 2[\lambda(\delta + \frac{1}{\delta})\tau + \mu(\delta - \frac{1}{\delta})\tau - 2\lambda]t \right\} \quad (46)$$

and the time when thermal fluctuations overtake quantum fluctuations is given by

$$t_d = \frac{\hbar\omega}{4kT[\lambda(\delta + \frac{1}{\delta})\tau + \mu(\delta - \frac{1}{\delta})\tau]}. \quad (47)$$

For  $\delta = 1$  we obtain from Eq. (46)

$$\sigma(t) = U(t) = \frac{\hbar^2}{4} \{ 1 + 4\lambda(\tau - 1)t \}, \quad (48)$$

independent of  $\mu$  and

$$t_d = \frac{\hbar\omega}{8kT\lambda}. \quad (49)$$

This particular result coincides with those obtained in Refs. [4, 5].

(d)  $r \neq 0$  : In the rest of this Section we analyze the generalized uncertainty function  $\sigma$  in the case when the correlation coefficient  $r$  is different from 0. At short times ( $\lambda t \ll 1, \Omega t \ll 1$ ), we obtain from Eq. (29):

$$\sigma(t) = \frac{\hbar^2}{4} \left\{ 1 + 2[\lambda(\delta + \frac{1}{\delta(1-r^2)}) \coth \epsilon + \mu(\delta - \frac{1}{\delta(1-r^2)}) \coth \epsilon - 2\lambda]t \right\}. \quad (50)$$

The time when thermal fluctuations become comparable with quantum fluctuations is in this case

$$t_d = \frac{1}{2[\lambda(\delta + \frac{1}{\delta(1-r^2)}) \coth \epsilon + \mu(\delta - \frac{1}{\delta(1-r^2)}) \coth \epsilon - 2\lambda]}. \quad (51)$$

(i) At zero temperature  $T = 0$ , the uncertainty becomes ( $\mu = 0$ ):

$$\sigma_0(t) = \frac{\hbar^2}{4} \left\{ 1 + 2\lambda(\delta + \frac{1}{\delta(1-r^2)} - 2)t \right\} \quad (52)$$

and

$$t_d = \frac{1}{2\lambda(\delta + \frac{1}{\delta(1-r^2)} - 2)}. \quad (53)$$

(ii) At high temperature

$$\sigma(t) = \frac{\hbar^2}{4} \left\{ 1 + 2[\lambda(\delta + \frac{1}{\delta(1-r^2)})\tau + \mu(\delta - \frac{1}{\delta(1-r^2)})\tau - 2\lambda]t \right\} \quad (54)$$

and the time when thermal fluctuations overtake quantum fluctuations is given by

$$t_d = \frac{\hbar\omega}{4kT[\lambda(\delta + \frac{1}{\delta(1-r^2)})\tau + \mu(\delta - \frac{1}{\delta(1-r^2)})\tau]}. \quad (55)$$

In summary of this section, the third main result is represented by: (1) the expressions (39) of the uncertainty  $\sigma = U$  (for  $r = 0$ ) and (50) of the uncertainty  $\sigma$  (for  $r \neq 0$ ) for short initial times, which evidently fulfil the uncertainty principle by virtue of the condition (22); (2) the corresponding expressions (40) and (51) for the time when the thermal fluctuations become comparable with the quantum fluctuations; we notice that this time is decreasing with the increasing of both temperature  $T$  and dissipation  $\lambda$ .

## 5 Discussion of results

One often regards the regime where thermal fluctuations begin to surpass quantum fluctuations as the transition point from quantum to classical statistical mechanics and identifies the high temperature regime of a system as the classical regime. On the other hand, it is known that a necessary condition for a system to behave classically is that the interference terms in its wave function have to diminish below a certain level, so that probability can be assigned to classical events [4, 5]. This is the decoherence process. The decoherence via interaction with an environment views the disappearance of the off-diagonal components of a reduced density matrix in some special basis as signaling a transition from quantum to classical physics. In Refs. [1, 2, 4, 5] it was shown that these two criteria of classicality are equivalent: the time when the quantum system decoheres is also the time when thermal fluctuations overtake quantum fluctuations. However the regime after thermal fluctuations dominate should not be called classical. After the decoherence time, although the system is describable in terms of probabilities, it cannot yet be regarded as classical because of the spin-statistics effects and has to be described by non-equilibrium quantum statistical mechanics. Only after the relaxation time the system can be correctly described by the equilibrium quantum statistical mechanics. The classical regime starts at a much later time. Only at a sufficiently high temperature when the spin (Fermi-Dirac or Bose-Einstein) statistics can be represented by the Maxwell-Boltzmann distribution function, can the system be considered in a classical regime [4, 5] (see Eqs. (36) – (38)).

The case of zero coupling,  $\lambda = 0$  and  $\mu = 0$ , corresponds to an isolated harmonic oscillator taken as a closed quantum system. We find the Heisenberg quantum

uncertainty function for an initial squeezed state to be (see Eq. (26))

$$U(t) = \frac{\hbar^2}{4} \left\{ 1 + \frac{1}{4} (\delta - \frac{1}{\delta})^2 \sin^2(2\omega t) \right\} \geq \frac{\hbar^2}{4}. \quad (56)$$

This is the quantum uncertainty relation for squeezed states. As the coupling to the environment goes to 0, the thermal fluctuations go also to 0 and the time-dependent term is the result of quantum fluctuations only. For the unsqueezed coherent state,  $\delta = 1$ , we recover the Heisenberg uncertainty relation  $U(t) = \hbar^2/4$ . For the same case of zero coupling, the Schrödinger uncertainty function (29) becomes  $\sigma(t) = \hbar^2/4$  for any correlated coherent initial state.

In all expressions for the uncertainty functions obtained in the preceding Sec. 4, the terms depending on  $t$  are functions of the initial spread and correlation coefficient and represent the initial growth of thermal fluctuations, starting from the pure quantum fluctuations at  $t = 0$ . Using condition (22), we notice that for short initial times the uncertainty increases with dissipation  $\lambda$  and temperature  $T$ . This is in contrast with other models studied in literature [1, 2, 4, 5], where the uncertainty principle is violated on a short time scale as a consequence of the well-known violation of the positivity of the density operator [2, 20, 21, 22]. For instance, in the model considered in Ref. [2], in the Fokker-Planck (high temperature) limit the uncertainty principle is violated for times  $t < (\hbar^2/\gamma k^2 T^2)^{1/3}$  in the case of uncertainty  $\sigma$  and for times  $t < \hbar/kT$  in the case of uncertainty  $U$ . Likewise, the uncertainty function  $U$  obtained in Refs. [4, 5] for high temperature  $T$  satisfies the uncertainty principle only for  $t > \hbar/kT$ . In addition, for a finite temperature there exists a restriction on how low the temperature could become for a given squeezing parameter, in order that the uncertainty principle to be satisfied [5]. In fact, the uncertainty principle in our model is fulfilled not only for short times, but for any time and temperature and for the full range of the squeezing and correlation parameters. Indeed, cf. Eq. (9), by virtue of the complete positivity property of the Lindblad model, the generalized uncertainty function  $\sigma$  always fulfills the uncertainty principle and for  $U(\geq \sigma)$  this is also true.

To exemplify the evolution of the uncertainty functions, in Fig. 1 we represent the Heisenberg  $U$  and Schrödinger  $\sigma$  uncertainties as functions on time and on temperature via  $\coth \epsilon$ , for  $\mu = 0, r = 0$  and for a given value of the squeezing parameter. In Fig. 2 the same uncertainties are represented as functions on time and on squeezing parameter, for  $\mu = 0, r = 0$  and for a finite temperature of the environment. In Fig. 3 the uncertainties are represented also as functions on time and on temperature for  $r = 0$  and for given values of the squeezing parameter  $\delta$  and  $\mu \neq 0$ . In Fig. 4 we

represent the dependence of the uncertainty function  $\sigma$  on time, on temperature and on the squeezing parameter for given finite values of  $r$  and  $\mu$ .

The time dependence of the uncertainties  $U(t)$  and  $\sigma(t)$  given by the dissipative terms reflects the fact that the Lindblad evolution of the system is non-unitary and is an expression of the effect of the environment. This is in contrast with the usual Liouville-von Neumann unitary evolution, when the uncertainty is independent of time, being invariant under unitary transformations.

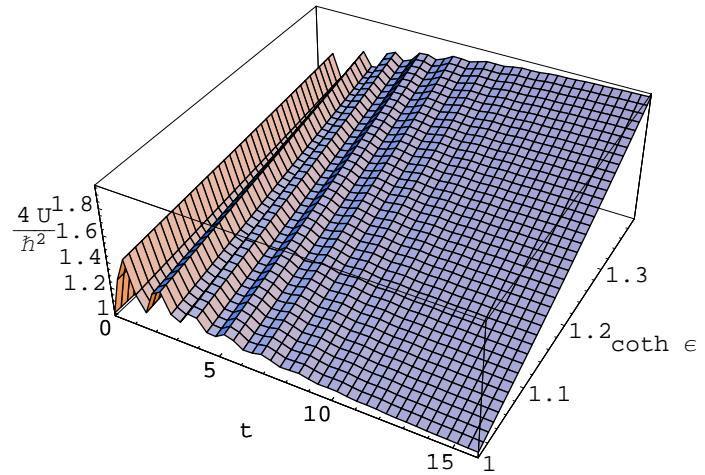
The limiting case  $\lambda = \mu$  deserves special attention in the Lindblad theory. As we mentioned in Sec. 2, the translational invariance is fulfilled in this case, but the asymptotic state is not a thermal state. Formally, the master equations considered by Halliwell and Hu [1, 2, 4, 5] can be obtained by taking  $D_{qq} = 0$  in the Lindblad master equation. In Lindblad theory this choice is forbidden, since in this case the fundamental constraints (4) are not any more fulfilled and this is the fact connected with the violation of the positivity of the density matrix in the mentioned models.

The time  $t_d$  when thermal fluctuations overtake quantum fluctuations obtained in our model for different initial states of the open system (correlated coherent states, squeezed states, coherent states) has in general a different form compared to the corresponding time obtained in Refs. [1, 2, 4, 5]. In the high temperature limit it is of the same scale as the decoherence time (time when the off-diagonal components of the density matrix responsible for interference decrease to zero due to the interaction with the environment). The value of this time was determined in a series of papers in quantum Brownian motion models for initial coherent states [23, 24, 25, 26], but not yet in the Lindblad model.

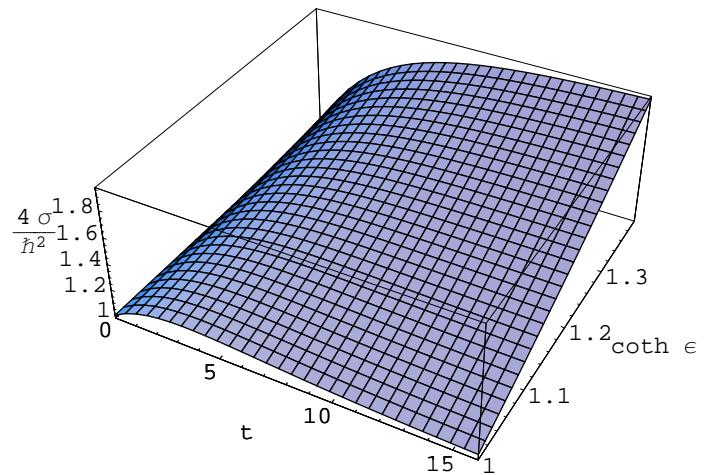
The second time scale of importance is the relaxation time scale,  $t_{rel} = \lambda^{-1} \gg t_d$ , when the particle reaches equilibrium with the environment. After this time, the uncertainty function takes on the Bose-Einstein form (36). At high temperatures the system reaches the Maxwell-Boltzmann limit and the uncertainty function takes on the classical form (38).

In the case of zero temperature, there are no longer thermal fluctuations and the environmentally induced fluctuations are of quantum nature only, given by terms describing both decay and oscillatory behaviour in the case of the uncertainty function  $U$  (28) and only decay behaviour in the case of the uncertainty function  $\sigma$  (35).

Many of the present results are related to those obtained in the cited papers [1, 2, 4, 5] and they lead essentially to the same physical conclusions in what concerns the role of thermal and quantum fluctuations in the transition from quantum to clas-

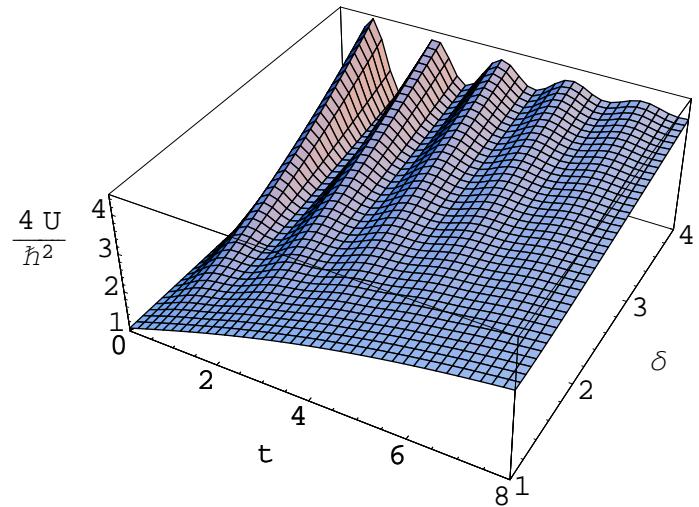


a)

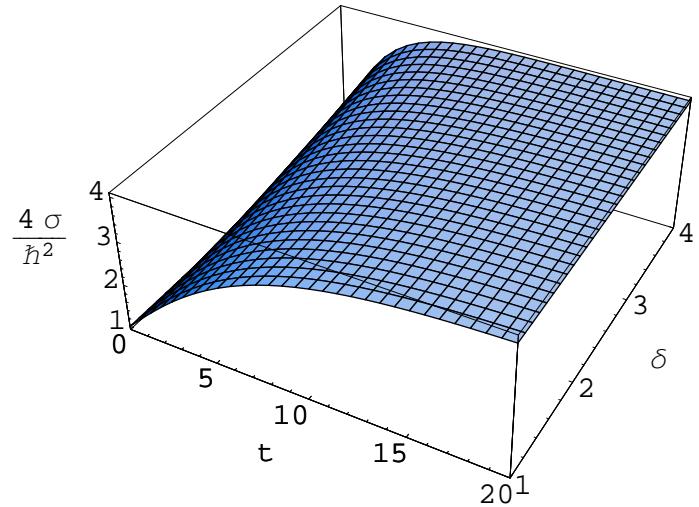


b)

Figure 1: a) Dependence of the Heisenberg uncertainty  $U$  (26) on time  $t$  and on temperature  $T$  via  $\coth \epsilon$  ( $\epsilon \equiv \coth(\hbar\omega/2kT)$ ), for  $\omega = 1$ ,  $\lambda = 0.1$ ,  $\mu = 0$ ,  $r = 0$  and  $\delta = 2$ . b) Dependence of the Schrödinger uncertainty  $\sigma$  (33) on the same variables and for the same values of the parameters. The unit of time is  $s$ .

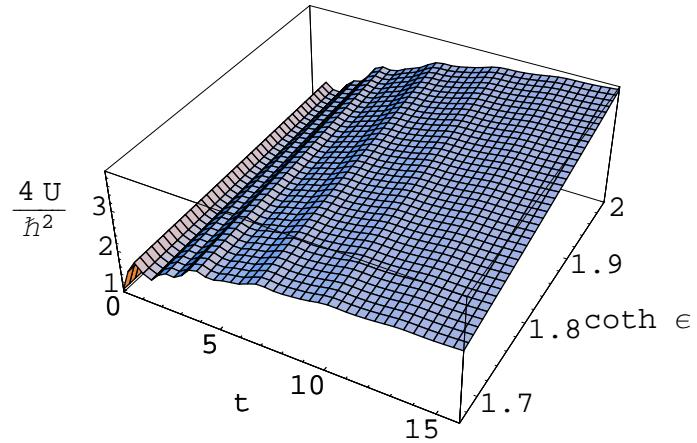


a)

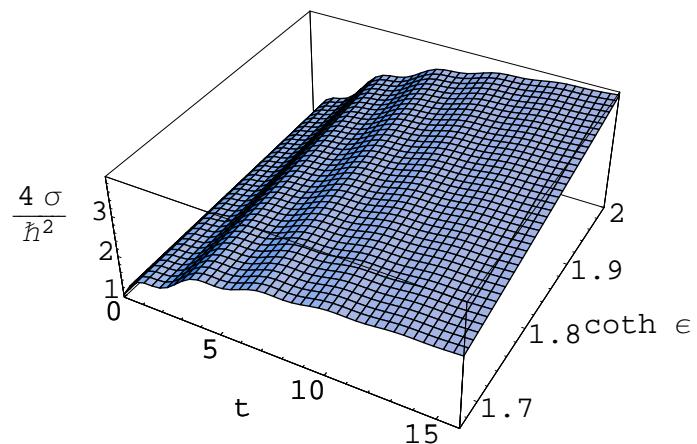


b)

Figure 2: a) Dependence of the Heisenberg uncertainty  $U$  (26) on time  $t$  and on the squeezing parameter  $\delta$  for  $\omega = 1$ ,  $\lambda = 0.1$ ,  $\mu = 0$ ,  $r = 0$  and  $\coth \epsilon = 2$ . b) Dependence of the Schrödinger uncertainty  $\sigma$  (33) on the same variables and for the same values of the parameters. The unit of time is  $s$ .

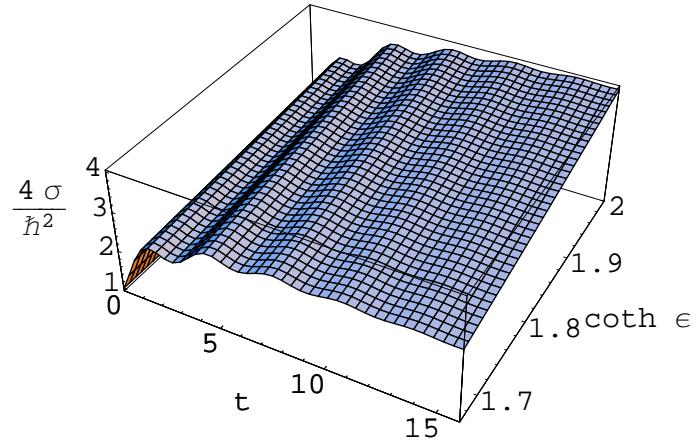


a)

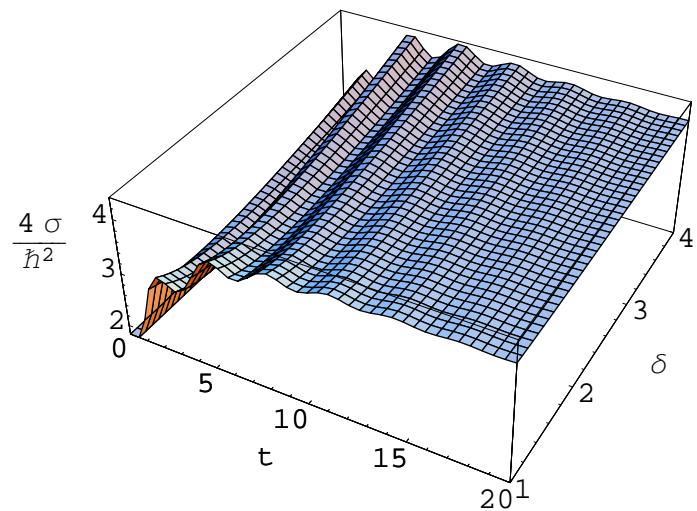


b)

Figure 3: a) Dependence of the Heisenberg uncertainty  $U$  (24) on time  $t$  and on temperature  $T$  via  $\coth \epsilon$  ( $\epsilon \equiv \coth(\hbar\omega/2kT)$ ), for  $\omega = 1$ ,  $\lambda = 0.1$ ,  $\mu = 0.08$ ,  $r = 0$  and  $\delta = 2$ . b) Dependence of the Schrödinger uncertainty  $\sigma$  (30) on the same variables and for the same values of the parameters. The unit of time is  $s$ .



a)



b)

Figure 4: a) Dependence of the Schrödinger uncertainty  $\sigma$  (29) on time  $t$  and on temperature  $T$  via  $\coth \epsilon$  ( $\epsilon \equiv \coth(\hbar\omega/2kT)$ ), for  $\omega = 1$ ,  $\lambda = 0.1$ ,  $\mu = 0.08$ ,  $r = 0.8$  and  $\delta = 2$ . b) Dependence of the same Schrödinger uncertainty  $\sigma$  on time  $t$  and on the squeezing parameter  $\delta$  for  $\omega = 1$ ,  $\lambda = 0.1$ ,  $\mu = 0.08$ ,  $r = 0.8$  and  $\coth \epsilon = 2$ . The unit of time is  $s$ .

sical behaviour, in concordance with the results of quantum mechanics, quantum and classical statistical mechanics. In this connection, we would like to add the following remarks concerning the differences between our paper and those earlier works.

(1) The uncertainty relations are studied in Refs. [1, 2, 4, 5] in the quantum Brownian motion model of an open system consisting of a harmonic oscillator linearly coupled to a thermal bath, using the Feynman-Vernon influence functional formalism to incorporate the statistical effect of the environment on the system in the reduced density matrix. In our paper, we use the theory of open quantum system based on the axiomatic approach of completely positive dynamical semigroups, which gives the most general form of a quantum Markovian master equation for the density operator of the open system.

(2) As is well-known, in the theory based on quantum dynamical semigroups the density operator satisfies the necessary requirements of unitarity, Hermiticity and positivity, while in the quantum Brownian motion model the positivity of the reduced density matrix is violated at short time scale. As a consequence, the uncertainty relations obtained in Refs. [1, 2, 4, 5] for both Heisenberg  $U$  and Schrödinger  $\sigma$  uncertainties violate the uncertainty principle for short initial time, for all temperatures  $T$ , including the Fokker-Planck limit of high temperatures. The only relation which fulfills the uncertainty principle for all  $t$  is obtained in Ref. [2], with the expense that it contains the finite cutoff frequency introduced in the spectral density. As we already mentioned, in the Lindblad model both uncertainty functions  $U$  and  $\sigma$  satisfy the uncertainty principle for the whole range of time  $t$ , temperature  $T$ , dissipative constant  $\lambda$  and for any initial correlated coherent state determined by the squeezing parameter  $\delta$  and correlation coefficient  $r$ .

(3) In general, the expressions of the time on which the thermal fluctuations become comparable to the quantum ones are obtained in Refs. [1, 2, 4, 5] in the Fokker-Planck limit of high temperatures. For the case of general  $T$ , even in the only expression obtained in Refs. [4, 5] for this time, a restriction has to be imposed on how low  $T$  could become for a given squeezing parameter. The expressions from the present paper for the time on which thermal fluctuations overtake the quantum ones are obtained for any temperature and contain an explicit dependence on  $\lambda, \delta$  and  $r$ . They have a different form from those obtained in the cited works (only in the particular case of high temperatures and for an initial coherent state ( $\delta = 1, r = 0$ ), our result is identical to that obtained in Refs. [4, 5]) and in the high temperature limit they are comparable with the decoherence time calculated in decoherence studies of transitions

from quantum to classical behaviour.

## 6 Summary and concluding remarks

In the present paper we have studied the evolution of the one-dimensional harmonic oscillator with dissipation within the framework of the Lindblad theory for open quantum systems. We have considered the general case of an environment consisting of a thermal bath at an arbitrary temperature. The Gaussian correlated coherent, squeezed and Glauber coherent states were taken as initial states. We have derived closed analytical expressions of the Heisenberg and Schrödinger uncertainty functions for the evolution of the damped harmonic oscillator for different regimes of time and temperature, in particular in the limiting cases of both zero temperature and high temperature of the environment, in the limit of short times and long times and in the limit of zero coupling between the system and environment (isolated harmonic oscillator). Besides the dissipation constant these expressions give the explicit dependence on the squeezing parameter and the correlation coefficient. The obtained uncertainty functions show explicitly the contributions of quantum and thermal fluctuations of the system and environment. There are three contributions to the uncertainty [1]: (i) uncertainty which is intrinsic to quantum mechanics, expressed through the Heisenberg uncertainty principle (37), which is not dependent on the dynamics; (ii) uncertainty that arises due to the spreading or reassembly (the reverse of spreading) of the wave packet, which depends on the dynamics and may increase or decrease the uncertainty; (iii) uncertainty due to the coupling to a thermal environment, which has two components: dissipation and diffusion (this latter is responsible for the process of decoherence); this generally tends to increase the uncertainty as time evolves. In the Lindblad model the uncertainty relations are fulfilled, while in some other models considered in literature, the uncertainty relations are violated at some initial moments of time.

We have described the evolution of the system from a quantum pure state to a non-equilibrium quantum statistical state and to an equilibrium quantum statistical state and we have analyzed the relaxation process. We also found the regimes in which each type of fluctuations is important. The three stages are marked by the decoherence time and the relaxation time, respectively. The regime in which thermal fluctuations become comparable with the quantum fluctuations coincides with the regime in which the decoherence effects come into play. In other words, the system evolves from a quantum-dominated state to a thermal-dominated state in a time which is comparable with the decoherence time calculated in similar models in the context of transitions

from quantum to classical physics [1, 2, 4, 5].

With this study one can understand the relation between quantum, thermal and classical fluctuations. With the two characteristic times, namely the relaxation time and the decoherence time to be determined in further studies on environment-induced decoherence in the Lindblad model, one can give further contributions in describing the role of quantum and thermal fluctuations and, using the uncertainty relations, the transition from quantum to classical physics. In this context we have shown recently [3, 27] that in the Lindblad model the Schrödinger generalized uncertainty relation is minimized for all times for Gaussian pure initial states of the form of correlated coherent states for a special choice of the diffusion and dissipation coefficients. Such states are therefore the ones that suffer the least amount of noise and they are connected with the decoherence phenomenon [24, 25, 26, 27], being the most predictable and stable under the evolution in the presence of the environment.

Recently there is an increased interest in quantum Brownian motion as a paradigm of quantum open systems. The possibility of preparing systems in macroscopic quantum states leads to the problems of dissipation in tunneling and of loss of quantum coherence (decoherence), which are intimately related to the issue of the transition from quantum to classical physics. The Lindblad theory provides a selfconsistent treatment of damping as a general extension of quantum mechanics to open systems and opens more possibilities to study these problems than the usual models of quantum Brownian motion.

### Acknowledgments

This work has partly been supported by a grant of the Romanian Ministry of Education and Research, under contract No. A6/2001. A. I. gratefully acknowledges this financial assistance. Financial support and hospitality at the Institute of Theoretical Physics in Giessen during the stay of one of the authors (A.I.) are also gratefully acknowledged.

## References

- [1] A. Anderson and J. J. Halliwell, Phys. Rev. D **48** (1993) 2753
- [2] C. Anastopoulos and J. J. Halliwell, Phys. Rev. D **51** (1995) 6870
- [3] A. Isar, Fortschritte der Physik **47** (1999) 855
- [4] B. L. Hu and Y. Zhang, Mod. Phys. Lett. A **8** (1993) 3575

- [5] B. L. Hu and Y. Zhang, *Int. J. Mod. Phys. A* **10** (1995) 4537
- [6] E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, New York, 1976)
- [7] H. Spohn, *Rev. Mod. Phys.* **52** (1980) 569
- [8] A. Isar, A. Sandulescu, H. Scutaru, E. Stefanescu and W. Scheid, *Int. J. Mod. Phys. E* **3** (1994) 635
- [9] G. Lindblad, *Commun. Math. Phys.* **48** (1976) 119
- [10] G. Lindblad, *Rep. Math. Phys.* **10** (1976) 393
- [11] A. Sandulescu and H. Scutaru, *Ann. Phys. (N.Y.)* **173** (1987) 277
- [12] A. Isar, A. Sandulescu and W. Scheid, *J. Math. Phys.* **34** (1993) 3887
- [13] A. Isar, *Helv. Phys. Acta* **67** (1994) 436; **68** (1995) 225
- [14] A. Isar, A. Sandulescu and W. Scheid, *J. Phys. G - Nucl. Part. Phys.* **17** (1991) 385
- [15] A. Isar, W. Scheid and A. Sandulescu, *J. Math. Phys.* **32** (1991) 2128
- [16] A. Isar, A. Sandulescu and W. Scheid, *Int. J. Mod. Phys. B* **10** (1996) 2767
- [17] E. Schrödinger, *Ber. Kgl. Akad. Wiss. (Berlin, 1930)* p. 296
- [18] H. P. Robertson, *Phys. Rev* **35** (1930) 667A, **46** (1934)
- [19] V. V. Dodonov, E. V. Kurmyshev and V. I. Man'ko, *Phys. Lett. A* **79** (1980) 150
- [20] A. O. Caldeira and A. J. Leggett, *Physica A* **121** (1983) 587
- [21] A. O. Caldeira and A. J. Leggett, *Ann. Phys. (N. Y.)* **149** (1983) 374
- [22] V. Ambegaokar, *Ber. Bunsenges. Phys. Chem.* **95** (1991) 400
- [23] B. L. Hu, J. P. Paz and Y. Zhang, *Phys. Rev. D* **45** (1992) 2843
- [24] J. P. Paz, S. Habib and W. Zurek, *Phys. Rev. D* **47** (1993) 488
- [25] W. Zurek, S. Habib and J. P. Paz, *Phys. Rev. Lett.* **70** (1993) 1187

- [26] W. Zurek, *Prog. Theor. Phys.* **89** (1993) 281
- [27] A. Isar, A. Sandulescu and W. Scheid, *Phys. Rev. E* **60** (1999) 6371